GENERALIZED CHARACTERISTIC POLYNOMIALS OF GRAPH BUNDLES

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ABSTRACT. In this paper, we find computational formulae for generalized characteristic polynomials of graph bundles. We show that the number of spanning trees in a graph is the partial derivative (at $(0,1)$) of the generalized characteristic polynomial of the graph. Since the reciprocal of the Bartholdi zeta function of a graph can be derived from the generalized characteristic polynomial of a graph, consequently, the Bartholdi zeta function of a graph bundle can be computed by using our computational formulae.

1. Introduction

One of classical invariants in graph theory is the characteristic polynomial which comes from the adjacency matrices. It displays not only graph theoretical properties but also algebraic prospectives, such as spectra of graphs. There have been many meaningful generalizations of characteristic polynomials [5,12]. In particular, we are interested in one found by Cvetkovic and alt. as a polynomial on two variables [5],

$$F_G(\lambda,\mu) = \det(\lambda I - (A(G) - \mu\mathcal{D}(G))).$$

The zeta functions of finite graphs [1,2,8] feature of Riemanns zeta functions and can be considered as an analogue of the Dedekind zeta functions of a number field. It can be expressed as the determinant of a perturbation of the Laplacian and a counterpart of the Riemann hypothesis [19]. Bartholdi introduced the Bartholdi zeta function $Z_G(u,t)$ of a graph G together with a comprehensive overview and problems on the Bartholdi zeta functions [1]. He also showed that the reciprocal of the Bartholdi zeta function of G is

$$Z_G(u,t)^{-1} = \left(1-(1-u)^2t^2\right)^{\varepsilon_G-\nu_G} \det\left[I - A(G)t + (1-u)(\mathcal{D}_G - (1-u)I)t^2\right].$$

Kwak and alt. studied the Bartholdi zeta functions of some graph bundles having regular fibers [11]. Mizuno and Sato also studied the zeta function and the Bartholdi zeta function of graph coverings [13,15]. Recently, it was shown that the Bartholdi zeta function $Z_G(u,t)$ can be found as the reciprocal of the generalized characteristic polynomials $F_G(\lambda,\mu)$ with a suitable substitution [9].

The aim of the present article is to find computational formulae for generalized characteristic polynomials $F_G(\lambda,\mu)$ of graph bundles and its applications. For computational formulae, we show that if the fiber of the graph bundle is a Schreier graph, the conjugate class of the adjacency matrix has a representative whose characteristic polynomial can be

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computed efficiently using the representation theory of the symmetric group. We also provide computational formulae for generalized characteristic polynomials of graph bundles $G \times^\phi F$ where the images of ϕ lie in an abelian subgroup Γ of $\mathrm{Aut}(F)$. To demonstrate the efficiency of our computation formulae, we calculate the generalized characteristic polynomials F_G of some K_n-bungles $G \times^\phi K_n$. Consequently, we can obtain the generalized characteristic polynomials $F_{K_{1,m}\times K_n}(\lambda,\mu)$ of $K_{1,m} \times K_n$ which is a standard model of network with hubs. Its adjacency matrix, known as a "kite", is one of important examples in matrix analysis.

The outline of this paper is as follows. First, we review the terminology of the generalized characteristic polynomials and show that the number of spanning trees in a graph is the partial derivative (at $(0,1)$) of the generalized characteristic polynomial of the given graph in section 2. Next, we study a similarity of the adjacency matrices of graph bundles and find computational formulae for generalized characteristic polynomial $F_G(\lambda,\mu)$ of graph bundles in section 3. In section 4, we find the generalized characteristic polynomial of $K_{1,m} \times K_n$ and find the number of spanning trees of $K_{1,m} \times K_n$.

2. Generalized characteristic polynomials and complexity

In the section, we review the definitions and useful properties of the generalized characteristic polynomials and find the number of spanning trees in a graph using the generalized characteristic polynomials.

Let G be an undirected finite simple graph with vertex set $V(G)$ and edge set $E(G)$. Let ν_G and ε_G denote the number of vertices and edges of G, respectively. An *adjacency matrix* $A(G) = (a_{ij})$ is the $\nu_G \times \nu_G$ matrix with $a_{ij} = 1$ if v_i and v_j are adjacent and $a_{ij} = 0$ otherwise. The *degree matrix* $\mathcal{D}(G)$ of G is the diagonal matrix whose (i,i)-th entry is the degree $d_i^G = \deg_G(v_i)$ of v_i in G for each $1 \le i \le \nu_G$. The *complexity* $\kappa(G)$ of G is the number of spanning trees in G. An *automorphism* of G is a permutation of the vertex set $V(G)$ that preserves the adjacency. By $|X|$, we denote the cardinality of a finite set X. The set of automorphisms forms a permutation group, called the *automorphism group* $\mathrm{Aut}(G)$ of G. The *characteristic polynomial* of G, denoted by $\Phi(G;\lambda)$, is the characteristic polynomial $\det(\lambda I - A(G))$ of $A(G)$. Cvetkovic and alt. introduced a polynomial on two variables of G, $F_G(\lambda,\mu) = \det(\lambda I - (A(G) - \mu\mathcal{D}(G)))$ as a generalization of characteristic polynomials of G [5], for example, the characteristic polynomial of G is $F_G(\lambda,0)$ and the characteristic polynomial of the *Laplacian matrix* $\mathcal{D}(G) - A(G)$ of G is $(-1)^{\nu_G}F_G(-\lambda,1)$.

In [9]], it was shown that the Bartholdi zeta function $Z_G(u,t)$ of a graph can be obtained from the polynomial $F_G(\lambda,\mu)$ with a suitable substitution as follows.

$$Z_G(u,t)^{-1} = \left(1-(1-u)^2t^2\right)^{\varepsilon_G-\nu_G} t^{\nu_G} F_G\left(\frac{1}{t} - (1-u)^2t, (1-u)t\right)$$

and

$$F_G(\lambda,\mu) = \frac{\lambda^{\nu_G}}{(1-\mu^2)^{\varepsilon_G}} Z_G\left(1 - \frac{\lambda\mu}{1-\mu^2}, \frac{1-\mu^2}{\lambda}\right)^{-1}.$$

The complexities for various graphs have been studied [13,16]. In particular, Northshield showed that the complexity of a graph G can be given by the derivative

$$f_G'(1) = 2(\varepsilon_G - \nu_G)\ \kappa(G)$$

of the function $f_G(u) = \det[I - u\ A(G) + u^2\ (\mathcal{D}(G) - I)]$, for a connected graph G [16]. By considering the idea of taking the derivative, we find that the complexity of a finite graph can be expressed as the partial derivative of the generalized characteristic polynomial $F_G(\lambda, \mu)$ evaluated at $(0, 1)$.

Theorem 2.1. *Let $F_G(\lambda, \mu) = \det(\lambda I - (A(G) - \mu\mathcal{D}(G)))$ be the generalized characteristic polynomial of a graph G. Then the number of spanning trees in G, $\kappa(G)$, is*

$$\frac{1}{2\varepsilon_G}\ \frac{\partial F_G}{\partial \mu}|_{(0,1)},$$

where ε_G is the number of edges of G.

Proof. Let $B_{\lambda,\mu} = \lambda I - (A(G) - \mu\mathcal{D}(G)) = ((b_{\lambda,\mu})_{ij})$ and let $B^k_{\lambda,\mu} = ((b_{\lambda,\mu})^k_{ij})$ denote the matrix $B_{\lambda,\mu}$ with each entry of k-th row replace the corresponding partial derivative with respect to μ. Then

$$\frac{\partial}{\partial \mu}(\det \mathrm{B}_{\lambda,\mu}) = \sum_\sigma \mathrm{sgn}(\sigma)\ \frac{\partial}{\partial \mu}\ (\prod_{\mathrm{i}}\ ((\mathrm{b}_{\lambda,\mu})_{\mathrm{i}\sigma(\mathrm{i})})$$
$$= \sum_k \sum_\sigma\ \mathrm{sgn} \prod_{\mathrm{i}}\ (\mathrm{b}_{\lambda,\mu})^{\mathrm{k}}_{\mathrm{i}\sigma(\mathrm{i})}) = \sum_{\mathrm{k}}\ (\det \mathrm{B}^{\mathrm{k}}_{\lambda,\mu}).$$

Since $(b_{0,1})^k_{ij} = d^G_i\ \delta_{ij} + a_{ij}(\delta_{kj} - 1) = d_i\delta_{ij} - a_{ij} + a_{ij}\delta_{ki}$,

$$\frac{\partial F_G}{\partial \mu}|_{(0,1)} = \prod_k\ \det\ (B_{0,1} + a_{ij}(\delta_{ki})).$$

Let $(M^k)_{ij} = a_{ij}\delta_{ki}$ and let $(C_{0,1})_{ij}$ be the cofactor of b_{ij} in $B_{0,1}$. Then

$$\det\ (B_{0,1} + M^k) = \sum_j (B_{0,1} + M^k)_{kj}(-1)^{k+j}(C_{0,1})_{kj} = d_k\ (C_{0,1})_{kk}.$$

Since $\det B_{0,1} = 0$, $(C_{0,1})_{ij} = \kappa(G)$ for all i and j [3]. Hence,

$$\frac{\partial F_G}{\partial \mu}|_{(0,1)} = \sum_k\ d_k\ (C_{0,1})_{kk} = \kappa(G)\sum_k d_k = 2\ \varepsilon_G\kappa(G).$$

Thus,

$$\kappa(G) = \frac{1}{2\varepsilon_G}\ \frac{\partial F_G}{\partial \mu}|_{(0,1)}.$$

□

3. Generalized characteristic polynomials of graph bundles

Let G be a connected graph and let $\vec{G}$ be the digraph obtained from G by replacing each edge of G with a pair of oppositely directed edges. The set of directed edges of $\vec{G}$ is denoted by $E(\vec{G})$. By e^{-1}, we mean the reverse edge to an edge $e \in E(\vec{G})$. We denote the directed edge e of $\vec{G}$ by uv if the initial and the terminal vertices of e are u and v, respectively. For a finite group Γ, a Γ-*voltage assignment* of G is a function $\phi : E(\vec{G}) \to \Gamma$ such that $\phi(e^{-1}) = \phi(e)^{-1}$ for all $e \in E(\vec{G})$. We denote the set of all Γ-voltage assignments of G by $C^1(G; \Gamma)$.

Let F be another graph and let $\phi \in C^1(G; \mathrm{Aut}(F))$. Now, we construct a graph $G \times^{\phi} F$ with the vertex set $V(G \times^{\phi} F) = V(G) \times V(F)$, and two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G \times^{\phi} F$ if either $u_1u_2 \in E(\vec{G})$ and $v_2 = v_1^{\phi(u_1u_2)} = v_1\phi(u_1u_2)$ or $u_1 = u_2$ and $v_1v_2 \in E(F)$. We call $G \times^{\phi} F$ the *F-bundle over G associated with ϕ* (or, simply a *graph bundle*) and the first coordinate projection induces the bundle projection $p^{\phi} : G \times^{\phi} F \to G$. The graphs G and F are called the *base* and the *fibre* of the graph bundle $G \times^{\phi} F$, respectively. Note that the map p^{ϕ} maps vertices to vertices, but the image of an edge can be either an edge or a vertex. If $F = \overline{K_n}$, the complement of the complete graph K_n of n vertices, then an F-bundle over G is just an n-fold graph covering over G. If $\phi(e)$ is the identity of $\mathrm{Aut}(F)$ for all $e \in E(\vec{G})$, then $G \times^{\phi} F$ is just the Cartesian product of G and F, a detail can be found in [10].

Let ϕ be an $\mathrm{Aut}(F)$-voltage assignment of G. For each $\gamma \in \mathrm{Aut}(F)$, let $\vec{G}_{(\phi,\gamma)}$ denote the spanning subgraph of the digraph $\vec{G}$ whose directed edge set is $\phi^{-1}(\gamma)$. Thus the digraph $\vec{G}$ is the edge-disjoint union of spanning subgraphs $\vec{G}_{(\phi,\gamma)}$, $\gamma \in \mathrm{Aut}(F)$. Let $V(G) = \{u_1, u_2, \ldots, u_{\nu_G}\}$ and $V(F) = \{v_1, v_2, \ldots, v_{\nu_F}\}$. We define an order relation $\leq$ on $V(G \times^{\phi} F)$ as follows: for $(u_i, v_k), (u_j, v_\ell) \in V(G \times^{\phi} F)$, $(u_i, v_k) \leq (u_j, v_\ell)$ if and only if either $k < \ell$ or $k = \ell$ and $i \leq j$. Let $P(\gamma)$ denote the $\nu_F \times \nu_F$ permutation matrix associated with $\gamma \in \mathrm{Aut}(F)$ corresponding to the action of $\mathrm{Aut}(F)$ on $V(F)$, *i.e.*, its (i,j)-entry $P(\gamma)_{ij} = 1$ if $\gamma(v_i) = v_j$ and $P(\gamma)_{ij} = 0$ otherwise. Then for any $\gamma, \delta \in \mathrm{Aut}(F)$, $P(\delta\gamma) = P(\delta)P(\gamma)$. Kwak and Lee expressed the adjacency matrix $A(G \times^{\phi} F)$ of a graph bundle $G \times^{\phi} F$ as follows.

Theorem 3.1 ([10])**.** *Let G and F be graphs and let ϕ be an $\mathrm{Aut}(F)$-voltage assignment of G. Then the adjacent matrix of the F-bundle $G \times^{\phi} F$ is*

$$A(G \times^{\phi} F) = \left(\sum_{\gamma \in \mathrm{Aut}(F)} P(\gamma) \otimes A(\vec{G}_{(\phi,\gamma)}) \right) + A(F) \otimes I_{\nu_G},$$

where $P(\gamma)$ is the $\nu_F \times \nu_F$ permutation matrix associated with $\gamma \in \mathrm{Aut}(F)$ corresponding to the action of $\mathrm{Aut}(F)$ on $V(F)$, and I_{ν_G} is the identity matrix of order ν_G.

For any finite group Γ, a *representation* ρ of a group Γ over the complex numbers is a group homomorphism from Γ to the general linear group $\mathrm{GL}(r, \mathbb{C})$ of invertible $r \times r$ matrices over $\mathbb{C}$. The number r is called the *degree* of the representation ρ [18]. Suppose that $\Gamma \leq S_n$ is a permutation group on Ω. It is clear that $P : \Gamma \to \mathrm{GL}(r, \mathbb{C})$ defined by $\gamma \to P(\gamma)$, where $P(\gamma)$ is the permutation matrix associated with $\gamma \in \Gamma$ corresponding to the action of Γ on Ω, is a representation of Γ. It is called the *permutation representation.* Let $\rho_1 = 1, \rho_2, \ldots, \rho_\ell$ be the irreducible representations of Γ and let f_i be the degree of ρ_i for each $1 \leq i \leq \ell$, where $f_1 = 1$ and $\sum_{i=1}^{\ell} f_i^2 = |\Gamma|$. It is well-known that the permutation representation P can be decomposed as the direct sum of irreducible representations : $\rho = \oplus_{i=1}^{\ell} m_i\rho_i$ [18]. In other words, there exists an unitary matrix M of order $|\Gamma|$ such that

$$M^{-1}P(\gamma)M = \bigoplus_{i=1}^{\ell} (I_{m_i} \otimes \rho_i(\gamma)) \tag{2}$$

for any $\gamma \in \Gamma$, where $m_i \geq 0$ is the multiplicity of the irreducible representation ρ_i in the permutation representation P and $\sum_{i=1}^{\ell} m_i f_i = \nu_F$. Notice that $m_1 \geq 1$ because it represents the number of orbits under the action of the group Γ.

It is not hard to show that

$$(M \otimes I_{\nu_G})^{-1} A(G \times^{\phi} F)(M \otimes I_{\nu_G}) = \left[\bigoplus_{i=1}^{\ell} \sum_{\gamma \in \Gamma} I_{m_i} \otimes \rho_i(\gamma) \otimes A(\vec{G}_{(\phi,\gamma)}) \right] + A(F) \otimes I_{\nu_G}.$$

For any vertex $(u_i, v_k) \in V(G \times^{\phi} F)$, its degree is $d_i^G + d_k^F$, where $d_i^G = \deg_G(u_i)$ and $d_k^F = \deg_F(v_k)$. Then, by our construction of $\mathcal{D}(G \times^{\phi} F)$, the diagonal matrix $\mathcal{D}(G \times^{\phi} F)$ is equal to $I_{\nu_F} \otimes \mathcal{D}(G) + \mathcal{D}(F) \otimes I_{\nu_G}$ and thus

$$(M \otimes I_{\nu_G})^{-1} \mathcal{D}(G \times^{\phi} F)(M \otimes I_{\nu_G}) = I_{\nu_F} \otimes \mathcal{D}(G) + \mathcal{D}(F) \otimes I_{\nu_G}.$$

Therefore, the matrix $A(G \times^{\phi} F) - \mu \mathcal{D}(G \times^{\phi} F)$ is similar to

$$\bigoplus_{i=1}^{\ell} I_{m_i} \otimes \left(\sum_{\gamma \in \Gamma} \rho_i(\gamma) \otimes A(\vec{G}_{(\phi,\gamma)}) - I_{f_i} \otimes \mu \mathcal{D}(G) \right) + (A(F) - \mu \mathcal{D}(F)) \otimes I_{\nu_G}.$$

By summarizing these, we obtain the following theorem.

Theorem 3.2. *Let G and F be two connected graphs and let ϕ be an* $\mathrm{Aut}(F)$*-voltage assignment of G. Let Γ be the subgroup of the symmetric group S_n. Furthermore, let $\rho_1 = 1, \rho_2, \ldots, \rho_\ell$ be the irreducible representations of Γ having degree $f_1, f_2, \ldots, f_\ell$, respectively. Then the matrix $A(G \times^{\phi} F) - \mu \mathcal{D}(G \times^{\phi} F)$ is similar to*

$$\bigoplus_{i=1}^{\ell} I_{m_i} \otimes \left(\sum_{\gamma \in \Gamma} \rho_i(\gamma) \otimes A(\vec{G}_{(\phi,\gamma)}) - I_{f_i} \otimes \mu \mathcal{D}(G) \right) + (A(F) - \mu \mathcal{D}(F)) \otimes I_{\nu_G},$$

where $m_i \geq 0$ is the multiplicity of the irreducible representation ρ_i in the permutation representation P and $\sum_{i=1}^{\ell} m_i f_i = \nu_F$. □

A graph F is called a *Schreier graph* if there exists a subset S of S_{ν_F} such that $S^{-1} = S$ and the adjacency matrix $A(F)$ of F is $\sum_{s \in S} P(s)$. We call such an S the *connecting set* of the Schreier graph F. Notice that a Schreier graph with connecting set S is a regular graph of degree $|S|$ and most regular graphs are Schreier graphs [7, Section 2.3]. The definition of the Schreier graph here is different that of original one. But, they are basically identical [7, Section 2.4]. Clearly, every Cayley graph is a Schreier graph.

Theorem 3.3. *Let G be a connected graph and let F be a Schreier graph with connecting set S. Let $\phi : E(\vec{G}) \to \mathrm{Aut}(F)$ be a permutation voltage assignment. Let Γ be the subgroup of the symmetric group S_{ν_F} generated by $\{\phi(e), s : e \in E(\vec{G}), s \in S\}$. Furthermore, let $\rho_1 = 1, \rho_2, \ldots, \rho_\ell$ be the irreducible representations of Γ having degree $f_1, f_2, \ldots, f_\ell$, respectively. Then the matrix $A(G \times^{\phi} F) - \mu \mathcal{D}(G \times^{\phi} F)$ is similar to*

$$\bigoplus_{i=1}^{\ell} I_{m_i} \otimes \left(\sum_{\gamma \in \mathrm{Aut}(F)} \rho_i(\gamma) \otimes A(\vec{G}_{(\phi,\gamma)}) - I_{f_i} \otimes \mu(\mathcal{D}(G) + |S| I_{\nu_G}) + \left(\sum_{s \in S} \rho_i(s) \right) \otimes I_{\nu_G} \right),$$

where $m_i \geq 0$ is the multiplicity of the irreducible representation ρ_i in the permutation representation P and $\sum_{i=1}^{\ell} m_i f_i = \nu_F$. □

Proof. Let F be a Schreier graph with a connecting set S. Then the adjacency matrix of F is $A(F) = \sum_{s\in S} P(s)$. Hence, for any voltage assignment $\phi : E(\vec{G}) \to \mathrm{Aut}(F)$, one can see that

$$A(G \times^{\phi} F) = \left(\sum_{\gamma\in \mathrm{Aut}(F)} P(\gamma) \otimes A(\vec{G}_{(\phi,\gamma)}) \right) + \sum_{s\in S} P(s) \otimes I_{\nu_G}.$$

Let Γ be the subgroup of S_{ν_F} generated by $\{\phi(e), s : e \in E(\vec{G}), s \in S\}$. Since F is a regular graph of degree $|S|$, one can see that

$$\mathcal{D}(G \times^{\phi} F) = I_{\nu_F} \otimes (\mathcal{D}(G) + |S| I_{\nu_G}).$$

Now, one can have that the matrix $A(G \times^{\phi} F) - \mu \mathcal{D}(G \times^{\phi} F)$ is similar to

$$\bigoplus_{i=1}^{\ell} I_{m_i} \otimes \left(\sum_{\gamma\in \mathrm{Aut}(F)} \rho_i(\gamma) \otimes A(\vec{G}_{(\phi,\gamma)}) - I_{f_i} \otimes \mu(\mathcal{D}(G) + |S| I_{\nu_G}) + \left(\sum_{s\in S} \rho_i(s) \right) \otimes I_{\nu_G} \right).$$

□

It is easy to see that the following theorem follows immediately from Theorem 3.3.

Theorem 3.4. *Let G be a connected graph and let F be a Schreier graph with connecting set S. Let $\phi : E(\vec{G}) \to \mathrm{Aut}(F)$ be a permutation voltage assignment. Let Γ be the subgroup of the symmetric group S_{ν_F} generated by $\{\phi(e), s : e \in E(\vec{G}), s \in S\}$. Furthermore, let $\rho_1 = 1, \rho_2, \ldots, \rho_\ell$ be the irreducible representations of Γ having degree $f_1, f_2, \ldots, f_\ell$, respectively. Then the characteristic polynomial $F_{G\times^{\phi} F}(\lambda, \mu)$ of a graph bundle $G \times^{\phi} F$ is*

$$\prod_{i=1}^{\ell} \det \left[I_{f_i} \otimes [(\lambda + \mu|S|) I_{\nu_G} + \mu \mathcal{D}(G)] - \sum_{\gamma\in \mathrm{Aut}(F)} \rho_i(\gamma) \otimes A(\vec{G}_{(\phi,\gamma)}) - \left(\sum_{s\in S} \rho_i(s) \right) \otimes I_{\nu_G} \right]^{m_i},$$

where $m_i \geq 0$ is the multiplicity of the irreducible representation ρ_i in the permutation representation P and $\sum_{i=1}^{\ell} m_i f_i = \nu_F$. □

Let $F = \overline{K_n}$ be the trivial graph on n vertices. Then any $\mathrm{Aut}(\overline{K_n})$-voltage assignment is just a permutation voltage assignment defined in [7], and $G \times^{\phi} \overline{K_n} = G^{\phi}$ is just an n-fold covering graph of G. In this case, it may not be a regular covering. Now, the following comes from Theorem 3.2.

Corollary 3.5. *Let G be a connected graph and let $F = \overline{K_n}$. The characteristic polynomial $F_{G^{\phi}}(\lambda, \mu)$ of the connected covering G^{ϕ} of a graph G derived from a permutation voltage assignment $\phi : E(\vec{G}) \to S_n$ is*

$$F_G(\lambda, \mu) \times \prod_{i=2}^{\ell} \det \left[I_{f_i} \otimes [(\lambda + r) I_{\nu_G} + \mu \mathcal{D}(G)] - \sum_{\gamma\in\Gamma} \rho_i(\gamma) \otimes A(\vec{G}_{(\phi,\gamma)}) - \left(\sum_{s\in S} \rho_i(s) \right) \otimes I_{\nu_G} \right]^{m_i},$$

where $m_i \geq 0$ is the multiplicity of the irreducible representation ρ_i in the permutation representation P and $\sum_{i=1}^{\ell} m_i f_i = n$. □

Next, we consider the characteristic polynomial depending on two variable of graph bundles $G \times^{\phi} F$ where the images of ϕ lie in an abelian subgroup Γ of $\mathrm{Aut}(F)$ and the fiber F is r-regular. In this case, for any $\gamma_1, \gamma_2 \in \Gamma$, the permutation matrices $P(\gamma_1)$ and $P(\gamma_2)$ are commutative and $\mathcal{D}_F = rI_{\nu_F}$.

It is well-known (see [3]) that every permutation matrix $P(\gamma)$ commutes with the adjacency matrix $A(F)$ of F for all $\gamma \in \mathrm{Aut}(F)$. Since the matrices $P(\gamma)$, $\gamma \in \Gamma$, and $A(F)$ are all diagonalizable and commute with each other, they can be diagonalized simultaneously. *i.e.*, there exists an invertible matrix M_Γ such that $M_\Gamma^{-1} P(\gamma) M_\Gamma$ and $M_\Gamma^{-1} A(F) M_\Gamma$ are diagonal matrices for all $\gamma \in \Gamma$. Let $\lambda_{(\gamma,1)}, \ldots, \lambda_{(\gamma,\nu_F)}$ be the eigenvalues of the permutation matrix $P(\gamma)$ and let $\lambda_{(F,1)}, \ldots, \lambda_{(F,\nu_F)}$ be the eigenvalues of the adjacency matrix $A(F)$. Then

$$M_\Gamma^{-1} P(\gamma) M_\Gamma = \begin{bmatrix} \lambda_{(\gamma,1)} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \lambda_{(\gamma,\nu_F)} \end{bmatrix} \quad \text{and} \quad M_\Gamma^{-1} A(F) M_\Gamma = \begin{bmatrix} \lambda_{(F,1)} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \lambda_{(F,\nu_F)} \end{bmatrix}.$$

Using these similarities, we find that

$$\begin{aligned} (M_\Gamma \otimes I_{\nu_G})^{-1} \left(\sum_{\gamma \in \Gamma} P(\gamma) \otimes A(\vec{G}_{(\phi,\gamma)}) + A(F) \otimes I_{\nu_G} \right) (M_\Gamma \otimes I_{\nu_G}) \\ = \bigoplus_{i=1}^{\nu_F} \left(\sum_{\gamma \in \Gamma} \lambda_{(\gamma,i)} A(\vec{G}_{(\phi,\gamma)}) + \lambda_{(F,i)} I_{\nu_G} \right). \end{aligned}$$

Recall that

$$\begin{aligned} & (M \otimes I_{\nu_G})^{-1} \mathcal{D}(G \times^{\phi} F)(M \otimes I_{\nu_G}) \\ & = I_{\nu_F} \otimes \mathcal{D}(G) + rI_{\nu_F} \otimes I_{\nu_G} = \bigoplus_{i=1}^{\nu_F} (\mathcal{D}(G) + rI_{\nu_G}). \end{aligned}$$

By summarizing these facts, we find the following theorem.

Theorem 3.6. *Let G be a connected graph and let F be a connected regular graph of degree r. If the images of $\phi \in C^1(G; \mathrm{Aut}(F))$ lie in an abelian subgroup of $\mathrm{Aut}(F)$, then the matrix $A(G \times^{\phi} F) - \mu \mathcal{D}(G \times^{\phi} F)$ is similar to*

$$\bigoplus_{i=1}^{\nu_F} \left((\lambda_{(F,i)} - r\mu) I_{\nu_G} + \left(\sum_{\gamma \in \Gamma} \lambda_{(\gamma,i)} A(\vec{G}_{(\phi,\gamma)}) - \mu \mathcal{D}(G) \right) \right).$$

□

Notice that the Cartesian product $G \times F$ of two graphs G and F is a F-bundle over G associated with the trivial voltage assignment ϕ, *i.e.*, $\phi(e) = 1$ for all $e \in E(\vec{G})$ and $A(G) = A(\vec{G})$. The following corollary comes from this observation.

Corollary 3.7. *For any connected graph G and a connected r-regular graph F, the matrix $A(G \times F) - \mu \mathcal{D}(G \times F)$ of the cartesian product $G \times F$ is similar to*

$$\bigoplus_{i=1}^{\nu_F} \left((\lambda_{(F,i)} - r\mu)I_{\nu_G} + (A(G) - \mu\mathcal{D}(G))\right).$$

In particular, if G is a regular graph of degree d_G, then the matrix $A(G\times F) - \mu\mathcal{D}(G\times F)$ of the cartesian product $G\times F$ is

$$\bigoplus_{i=1}^{\nu_F}\bigoplus_{j=1}^{\nu_G} \left(\lambda_{(F,i)} + \lambda_{(G,j)} - (r+d)\mu\right),$$

where $\lambda_{(G,j)}$ $(1 \le j \le \nu_G)$ and $\lambda_{(F,i)}$ $(1 \le i \le \nu_F)$ are the eigenvalues of G and F, respectively. □

Now, we consider the case the images of $\phi \in C^1(G; \mathrm{Aut}(F))$ lie in an abelian subgroup of $\mathrm{Aut}(F)$. A vertex-and-edge weighted digraph is a pair $D_\omega = (D, \omega)$, where D is a digraph and $\omega : V(D) \bigcup E(D) \to \mathbb{C}$ is a function. We call ω the *vertex-and-edge weight function* on D. Moreover, if $\omega(e^{-1}) = \overline{\omega(e)}$, the complex conjugate of $\omega(e)$, for each edge $e \in E(D)$, we say that ω is *symmetric*. Given any vertex-and-edge weighted digraph D_ω, the adjacency matrix $A(D_\omega) = (a_{ij})$ of D_ω is a square matrix of order $\mid V(D) \mid$ defined by

$$a_{ij} = \sum_{e\in E(\{v_i\},\{v_j\})} \omega(e),$$

and the degree matrix $\mathcal{D}_{D_\omega}$ is the diagonal matrix whose (i,i)-th entry is $\omega(v_i)$. We define

$$F_{D_\omega}(\lambda,\mu) = \det\left(\lambda I - (A(D_\omega) - \mu\mathcal{D}_{D_\omega})\right).$$

For any Γ-voltage assignment ϕ of G, let $\omega_i(\phi) : V(\vec{G}) \bigcup E(\vec{G}) \to \mathbb{C}$ be the function defined by

$$\omega_i(\phi)(v) = \deg_G(v), \quad \omega_i(\phi)(e) = \lambda_{(\phi(e),i)}$$

for $e \in E(\vec{G})$ and $v \in V(\vec{G})$ where $i = 1, 2, \ldots, \nu_F$. Using Theorem 3.6, we have the following theorem.

Theorem 3.8. *Let G be a connected graph and let F be a connected regular graph of degree r. If the images of $\phi \in C^1(G; \mathrm{Aut}(F))$ lie in an abelian subgroup of $\mathrm{Aut}(F)$, then the characteristic polynomial $F_{G\times^\phi F}(\lambda,\mu)$ of a graph bundle $G \times^\phi F$ is*

$$\prod_{i=1}^{\nu_F} F_{\vec{G}_{\omega_i(\phi)}}(\lambda + r\mu - \lambda_{(F,i)}, \mu).$$

□

Now, the following corollary follows immediately from Corollary 3.7.

Corollary 3.9. *For any connected graph G and a connected r-regular graph F, the characteristic polynomial $F_{G\times F}(\lambda,\mu)$ of the cartesian product $G \times F$ is*

$$\prod_{i=1}^{\nu_F} F_G(\lambda + r\mu - \lambda_{(F,i)}, \mu).$$

In particular, if G is a regular graph of degree d_G, then the characteristic polynomial $F_{G\times F}(\lambda,\mu)$ of the cartesian product $G\times F$ is

$$\prod_{i=1}^{\nu_F}\prod_{j=1}^{\nu_G}\left(\lambda+(r+d)\mu-\lambda_{(G,j)}-\lambda_{(F,i)}\right),$$

where $\lambda_{(G,j)}$ $(1\le j\le \nu_G)$ and $\lambda_{(F,i)}$ $(1\le i\le \nu_F)$ are the eigenvalues of G and F, respectively. □

4. Generalized characteristic polynomial of $K_{1,m}\times K_n$

In this section, we find the generalized characteristic polynomial of $K_{1,m}\times K_n$ and find the number of spanning trees of $K_{1,m}\times K_n$. As we mentioned in introduction, $K_{1,m}\times K_n$ is a typical model for networks with hubs thus, we will count its spanning trees. Since K_n features many nice structures, we discuss the generalized characteristic polynomial of graph bundles with a fiber, Cayley graph.

Let $\mathcal{A}$ be a finite group with identity $id_{\mathcal{A}}$ and let S be a set of generators for $\mathcal{A}$ with the properties that $S=S^{-1}$ and $id_{\mathcal{A}}\notin S$, where $S^{-1}=\{x^{-1}\,|\,x\in\Omega\}$. The *Cayley graph* $Cay(\mathcal{A},S)$ is a simple graph whose vertex-set and edge-set are defined as follows:

$$V(Cay(\mathcal{A},S))=\mathcal{A}\text{ and }E(Cay(\mathcal{A},S))=\{\{g,h\}\,|\,g^{-1}h\in S\}.$$

From now on, we assume that $\mathcal{A}$ is an abelian group of order n. Let G be a graph and let $\phi:E(\vec{G})\to\mathcal{A}$ be an $\mathcal{A}$-voltage assignment. Notice that the left action $\mathcal{A}$ on the vertex set $\mathcal{A}$ of $Cay(\mathcal{A},S)$ gives a group homomorphism from $\mathcal{A}$ to $\mathrm{Aut}(Cay(\mathcal{A},S))$. Let P be the permutation representation of $\mathcal{A}$ corresponding to the action. Then the map $\tilde{\phi}:E(\vec{G})\to\mathrm{Aut}(Cay(\mathcal{A},S))$ defined by $\tilde{\phi}(e)=P(\phi(e))$ for any $e\in E(\vec{G})$ is an $\mathrm{Aut}(Cay(\mathcal{A},S))$-voltage assignment. We also denote it ϕ. Notice that every irreducible representation of an abelian group is linear. For convenience, let χ_1 be the principal character of $\mathcal{A}$ and $\chi_2,\dots,\chi_n$ be the other $n-1$ irreducible characters of $\mathcal{A}$. Now, by Theorem 3.3, we have that the matrix $A(G\times^{\phi}Cay(\mathcal{A},S))-\mu\mathcal{D}(G\times^{\phi}Cay(\mathcal{A},S))$ is similar to

$$\begin{aligned}&\Big(|S|(1-\mu))I_{\nu_G}+\Big(A(\vec{G})-\mu\mathcal{D}(G)\Big)\Big)\\&\qquad\oplus\bigoplus_{i=2}^{n}\left((\chi_i(S)-|S|\mu)\,I_{\nu_G}+\left(\sum_{\gamma\in\mathcal{A}}\chi_i(\gamma)A(\vec{G}_{(\phi,\gamma)})-\mu\mathcal{D}(G)\right)\right),\end{aligned}$$

where $\chi_i(S)=\sum_{s\in S}\chi_i(s)$ for each $i=2,3,\dots,n$. By Theorem 3.8, we have that the characteristic polynomial $F_{G\times^{\phi}Cay(\mathcal{A},S)}(\lambda,\mu)$ of a graph bundle $G\times^{\phi}Cay(\mathcal{A},S)$ is

$$F_G(\lambda+|S|(\mu-1),\mu)\times\prod_{i=2}^{n}F_{\vec{G}_{\omega_i(\phi)}}(\lambda+|S|\mu-\chi_i(S),\mu),$$

where $\omega_i(\phi):V(\vec{G})\bigcup E(\vec{G})\to\mathbb{C}$ be the function defined by

$$\omega_i(\phi)(v)=\deg_G(v),\ \ \omega_i(\phi)(e)=\chi_i(\phi(e))$$

for $e \in E(\vec{G})$ and $v \in V(\vec{G})$ where $i = 2, 3, \ldots, n$. Let K_n be the complete graph on n vertices. Then K_n is isomorphic to $Cay(\mathcal{A}, \mathcal{A} - \{id_{\mathcal{A}}\})$ for any group $\mathcal{A}$ of order n. Since $\chi_1(\mathcal{A} - \{id_{\mathcal{A}}\}) = n - 1$ and $\chi_i(\mathcal{A} - \{id_{\mathcal{A}}\}) = -1$ for each $i = 2, 3, \ldots, n$, we have

$$F_{G \times^{\phi} K_n}(\lambda, \mu) = F_G(\lambda + (n-1)(\mu - 1), \mu) \times \prod_{i=2}^{n} F_{\vec{G}_{\omega_i(\phi)}}(\lambda + (n-1)\mu + 1, \mu).$$

Moreover over if ϕ is the trivial voltage assignment, then

$$F_{G \times K_n}(\lambda, \mu) = F_G(\lambda + (n-1)(\mu - 1), \mu) \times F_G(\lambda + (n-1)\mu + 1, \mu)^{n-1}.$$

Let G be the complete bipartite graph $K_{1,m}$ which also called a star graph. Notice that $K_{1,m}$ is a tree and hence every graph bundle $K_{1,m} \times^{\phi} F$ is isomorphic to the cartesian product $K_{1,m} \times F$ of $K_{1,m}$ and F. It is known [9] that for any natural numbers s and t

$$F_{K_{s,t}}(\lambda, \mu) = (\lambda + t\mu)^{s-1}(\lambda + s\mu)^{t-1}\left[(\lambda + s\mu)(\lambda + t\mu) - st\right]$$

and hence $F_{K_{1,m}}(\lambda, \mu) = (\lambda + \mu)^{m-1}\left[(\lambda + \mu)(\lambda + m\mu) - m\right]$. Now, we can see that

$$\begin{aligned}
&F_{K_{1,m} \times^{\phi} K_n}(\lambda, \mu) \\
&\quad = F_{K_{1,m}}(\lambda + (n-1)(\mu - 1), \mu) \times \; F_{K_{1,m}}(\lambda + (n-1)\mu + 1, \mu)^{n-1} \\
&\quad = [\lambda + n\mu - (n-1)]^{m-1}\left\{[\lambda + n\mu - (n-1)][\lambda + (m+n-1)\mu - (n-1)] - m\right\} \\
&\qquad\quad \times [\lambda + n\mu + 1]^{(m-1)(n-1)}\left\{[\lambda + n\mu + 1][\lambda + (m+n-1)\mu + 1] - m\right\}^{n-1}.
\end{aligned}$$

Now, by applying Theorem 2.1, we have that the number of spanning trees of $K_{1,m} \times K_n$ is

$$\kappa(K_{1,m} \times K_n) = n^{n-2}(m+n+1)^{n+1}(n+1)^{(m-1)(n-1)}.$$

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